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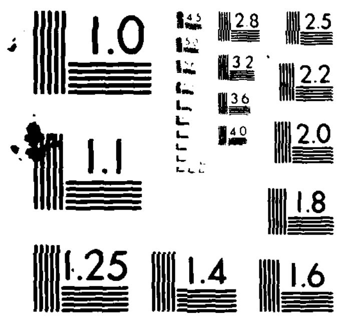

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NONLINEAR PARABOLIC EQUATIONS INVOLVING MEASURES AS INITIAL CONDITIONS

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ABSTRACT

We first consider the Cauchy problem for certain equations (is considered)

(1) $$u_t - \Delta u + |u|^{p-1} u = 0 \quad \text{on} \quad \Omega \times (0,T)$$

with a boundary condition and ~~the~~ an initial condition.

(2) $$u(x,0) = \delta(x) \quad \text{on} \quad \Omega$$

where Ω R^n is domain containing 0, $0 < p < \infty$, $0 < T < \infty$ and $\delta(x)$ is the Dirac mass at 0. We prove that a solution of ~~(1) - (2)~~ the equations exists if and only if $0 < p < \frac{n+2}{n}$. When $0 < p < \frac{n+2}{n}$ we actually prove a more general existence and uniqueness result in which (2) is replaced by

(3) $$u(x,0) = u_0(x) \quad \text{on} \quad \Omega$$

where u_0 is a measure.

Next, we discuss the Cauchy problem for

(4) $$u_t - \Delta(|u|^{m-1}u) = 0 \quad \text{on} \quad \Omega \times (0,T)$$

where $0 < m < \infty$, with a boundary condition and the initial condition (3). We prove that a solution of (4) - (2) exists if and only if $m > \frac{n-2}{n}$. When $m > \frac{n-2}{n}$ we actually prove existence for the problem (4) - (3).

AMS(MOS) Subject Classifications: 35K15, 35K55

Key Words: Nonlinear parabolic equations; Measures as initial conditions; Nonexistence; Boundary layer; Removable singularities; Porous media equation; Regularizing semigroups; Compact semigroups.

Work Unit Number 1 - Applied Analysis

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SIGNIFICANCE AND EXPLANATION

Nonlinear evolution equations of the form

$$u_t - \Delta u + |u|^{p-1}u = 0 \quad \text{on} \quad \mathbb{R}^n \times (0,T)$$

or

$$u_t - \Delta(|u|^{m-1}u) = 0 \quad \text{on} \quad \mathbb{R}^n \times (0,T)$$

arise in a large variety of problems in physics and mechanics. This paper deals with the question of existence (and uniqueness) when the initial data is a measure, for example a Dirac mass. Physically this corresponds to the important case when the initial temperature (or initial density etc. ..) is extremely high near one point. The main novelty of this paper is to show that a solution exists only under some severe restrictions on the parameter p (or m); namely p must be less than $\frac{n+2}{n}$ $(m > \frac{n-2}{n})$. For example, one striking conclusion reached is the fact that ~~the~~ an equation

$$(1) \qquad \begin{cases} u_t - \Delta u + u^3 = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ u(x,0) = \delta(x) \end{cases}$$

possesses no solution in any dimension $n \geq 1$ and on any time interval $(0,T)$. This result pinpoints the sharp contrast between linear and nonlinear equations from the point of view of existence. It also implies that linearization is meaningless for equations of the type (1) ever small time interval.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

NONLINEAR PARABOLIC EQUATIONS INVOLVING MEASURES AS INITIAL CONDITIONS

Haim Brezis and Avner Friedman

1. Introduction

In this paper we first consider the Cauchy problem for the nonlinear parabolic equation

(1) $$u_t - \Delta u + |u|^{p-1}u = 0 \quad \text{on} \quad \Omega \times (0,T)$$

with a boundary condition and the initial condition

(2) $$u(x,0) = \delta(x) \quad \text{on} \quad \Omega$$

where Ω R^n is a domain containing 0, $0 < p < \infty$, $0 < T < \infty$ and $\delta(x)$ denotes the Dirac mass at 0..

We prove that a solution of (1) - (2) exists <u>if and only if</u> $0 < p < \frac{n+2}{n}$. In particular the equation

$$u_t - \Delta u + u^3 = 0 \quad \text{on} \quad \Omega \times (0,T)$$

$$u(x,0) = \delta(x) \quad \text{on} \quad \Omega$$

has <u>no solution in any dimension</u> $n \geq 1$. We derive the <u>nonexistence</u> claim from a statement about "removable singularities"; we show that there is a <u>local</u> obstruction to the existence of a solution of (1) - (2) when $p \geq \frac{n+2}{n}$ no matter what conditions we impose on the boundary $\partial\Omega$. When $0 < p < \frac{n+2}{n}$ we actually prove a more general existence and uniqueness result in which (2) is replaced by

(3) $$u(x,0) = u_0(x) \quad \text{in} \quad \Omega$$

where $u_0(x)$ is a <u>measure</u>.

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Next we discuss the Cauchy problem for the equation

$$(4) \qquad u_t - \Delta(|u|^{m-1}u) = 0 \quad \text{on} \quad \Omega \times (0,T)$$

where $m > 0$, with a boundary condition and the initial condition (2). We prove that a solution of (4) - (2) exists <u>if and only if</u> $m > \frac{n-2}{n}$ (any $m > 0$ when $n = 1$ or 2). We actually prove an existence result for (4) - (3) when $m > \frac{n-2}{n}$.

The solvability of (4) - (3) when u_0 is a measure has been considered by various authors. If $\Omega = R^n$, $u_0(x) = \delta(x)$ and $m > \frac{n-2}{n}$, an explicit solution of (4) - (3) was given by Barenblatt [4] (see also Pattle [21]). If $\Omega = R^n$, $m > 1$, $u_0 \geq 0$ is a bounded measure, existence and uniqueness was obtained by M. Pierre [23], even for more general nonlinearities $\phi(u)$ - not just $|u|^{m-1}u$ [the case $n = 1$ had been treated earlier by S. Kamin [18]). The <u>non existence</u> aspect seems however to be new. Non existence results for (1) - (2) (or (4) - (2)) are somewhat surprizing in view of the following facts:

i) solutions of (1) - (3) [or (4) - (3)] are known to exist for <u>any</u> $u_0 \in L^1(\Omega)$ under <u>no restriction</u> on $p > 0$ (or $m > 0$)

ii) a priori estimates do not "distinguish" between L^1 functions and measures.

This apparent contradiction will be explained in Sections 3 and 4.

<u>Existence</u> and <u>non existence</u> results for <u>elliptic</u> equations of the form

$$-\Delta u + |u|^{p-1}u = f \quad \text{on} \quad \Omega$$

where f is a <u>measure</u> have been obtained by Bamberger [2], Benilan-Brezis [6] and Brezis-Veron [12]. Our approach borrows some ideas from these papers. The results concerning equation (1) are presented in Section 2, 3 and 4.

In Section 2 we prove non existence and removable singularities for (1) - (2) when $p \geq \frac{n+2}{n}$.

In Section 3 we prove existence and uniqueness of a solution of (1) - (3) when $p < \frac{n+2}{n}$.

In Section 4 we assume $p > \frac{n+2}{n}$ and we study the limiting behavior of a sequence u_j of solutions of (1) corresponding to a sequence of smooth initial data $u_{0j} \to \delta$. We exhibit a boundary layer phenomenon at $t = 0$; in the process of passing to the limit one loses the natural initial condition.

In Section 5 we discuss the properties of equation (4).

2. Non existence and removable singularities for equation (1) when $p > \frac{n+2}{n}$.

Let $\Omega \subset \mathbb{R}^n$ be any open set with $0 \in \Omega$. Assume $p > \frac{n+2}{n}$.

Definition. A solution of (1) is a function $u(x,t) \in L^p_{\ell oc}(\Omega \times (0,T))$ such that (1) holds in the sense of distributions i.e.

$$-\iint u\phi_t dxdt - \iint u\Delta\phi \, dxdt + \iint |u|^{p-1}u \, \phi dxdt = 0 \quad \forall \phi \in \mathcal{D}(\Omega \times (0,T)) \; .$$

The main results of Section 2 are the following

Theorem 1. There is no solution of (1) such that

$$\operatorname*{ess\,lim}_{t\to 0} \int u(x,t)\phi(x)dx = \phi(0) \quad \forall \phi \in C_c(\Omega)^{(1)}$$

Theorem 1 is an immediate consequence of

Theorem 2. Assume u is a solution of (1) such that

$$\operatorname*{ess\,lim}_{t\to 0} \int u(x,t)\phi(x)dx = 0 \quad \forall \phi \in C_c(\Omega\backslash\{0\}) \; . \tag{5}$$

Then $u \in C^{2,1}(\Omega \times [0,T))^{(2)}$ and $u(x,0) = 0$ on Ω.

Remark 1. Theorem 2 implies in particular the following. Let u be a classical solution of (1) on $\Omega \times (0,T)$. Assume that u is continuous on $\Omega \times [0,T)$ except possibly at the point $(x,t) = (0,0)$ and that $u(x,0) = 0$ on $\Omega\backslash\{0\}$. Conclusion: u has no singularity at $(0,0)$.

Note the sharp contrast with the behavior of solutions of linear parabolic equations. For example the fundamental solution $E(x,t)$ of the heat equation satisfies:

i) $E_t - \Delta E = 0$ in $\mathbb{R}^n \times (0,T)$

ii) $E(x,t)$ is smooth on $\mathbb{R}^n \times [0,T)$ except at the point $(x,t) = (0,0)$ and $E(x,0) = 0$ for $x \neq 0$

(1) $C_c(\Omega)$ denotes the space of all continuous functions with compact support in Ω.

(2) $C^{2,1}$ denotes the space of all continuous functions $u(x,t)$ having continuous derivatives u_t, u_{x_i}, $u_{x_i x_j}$.

iii) E has a singularity at $(0,0)$.

Remark 2. In Theorem 2 one may replace condition (5) by the weaker condition

$$\text{(5')} \qquad \operatorname*{ess\,lim}_{t\to 0} \int u(x,t)\phi(x)dx = 0 \quad \forall \phi \in \mathcal{D}(\Omega\setminus\{0\})$$

provided $u \geq 0$ (because, in that case, (5) <==> (5')). However if u changes sign we don't know whether the conclusion of Theorem 2 is still valid under the assumption (5').

The proof of Theorem 2 is divided into 6 steps. In what follows u denotes a solution of (1) satisfying (5).

Step 1. We have $u \in C^{2,1}(\Omega \times (0,T))$.

Proof. We shall use a parabolic version of Kato's inequality.

Lemma 1. Let $Q \subset \mathbb{R}^n \times \mathbb{R}$ be any open set. Let $u \in L^1_{\ell oc}(Q)$ be such that

$$u_t - \Delta u = f \quad \text{in} \quad \mathcal{D}'(Q)$$

with $f \in L^1_{\ell oc}(Q)$. Then

$$|u|_t - \Delta|u| \leq f \text{ sign } u \quad \text{in} \quad \mathcal{D}'(Q) \quad .^{(1)}$$

Since the proof is almost identical to the proof in the elliptic case (see Kato [19]) we shall omit it.

From (1) and Lemma 1 we deduce that

$$\text{(6)} \qquad |u|_t - \Delta|u| + |u|^p \leq 0 \quad \text{in} \quad \mathcal{D}'(\Omega \times (0,T))$$

and in particular

$$\text{(7)} \qquad |u|_t - \Delta|u| \leq 0 \quad \text{in} \quad \mathcal{D}'(\Omega \times (0,T)) \quad .$$

Therefore $|u|$ is subcaloric in $\Omega \times (0,T)$ and consequently $u \in L^\infty_{\ell oc}(\Omega \times (0,T))$. Indeed a mollifier U_ε of $|u|$ still satisfies (7). Representing it in terms of Green's function in a cube K_r with sides

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(1)

$$\text{sign } u = \begin{cases} 1 & \text{if} \quad u > 0 \\ 0 & \text{if} \quad u = 0 \\ -1 & \text{if} \quad u < 0 \end{cases}$$

parallel to the axes we obtain (see Friedman [17] p. 130)

$$U_\varepsilon(x,t) \le C_r \int_{\partial_p K_r} U_\varepsilon$$

where $\partial_p K_r$ is the parabolic boundary of K_r and (x,t) is the center of its top face. Integrating with respect to r in some interval $0 < r_1 < r < r_2$ and taking $\varepsilon \to 0$ we obtain that $u \in L^\infty_{\ell oc}(\Omega \times (0,T))$.

Using (1) and the standard regularity theory for the heat equation we conclude that $u \in C^{2,1}(\Omega \times (0,T))$. In fact, u is as smooth as the function $u \mapsto |u|^{p-1}u$ permits. In particular if p is an integer then $u \in C^\infty(\Omega \times (0,T))$.

<u>Step 2</u>. Let $\omega \subset\subset \Omega\backslash\{0\}$(1). Fix $T_1 < T$. Then we have

(8) $$u \in L^\infty(0,T_1; L^1(\omega))$$

(9) $$u \in L^p(0,T_1; L^p(\omega)) \quad .$$

<u>Proof of (8)</u>. Suppose by contradiction that for a sequence t_n in $(0,T_1)$, $\|u(\cdot,t_n)\|_{L^1(\omega)} \to \infty$.

Since $u \in L^\infty_{\ell oc}(\Omega \times (0,T))$ we have $t_n \to 0$. On the other hand, we deduce from (5) and the uniform boundedness principle that $\|u(\cdot,t_n)\|_{L^1(\omega)}$ remains bounded as $t_n \to 0$.

<u>Proof of (9)</u>. Let $\zeta \in \mathcal{D}(\Omega\backslash\{0\})$ be such that $0 \le \zeta \le 1$, $\zeta = 1$ on ω.

From (6) we deduce that for $0 < \varepsilon < T_1$

(10)
$$\int |u(x,T_1)|\zeta(x)dx + \int_\varepsilon^{T_1} \int |u(x,t)|^p\zeta(x)dxdt \le$$
$$\le \int |u(x,\varepsilon)|\zeta(x)dx + \int_\varepsilon^{T_1} \int |u(x,t)|\Delta\zeta(x)dx \quad .$$

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(1) As usual this notation means that ω is an open set such that $\bar\omega \subset \Omega\backslash\{0\}$.

From (8) we know that the right hand side in (10) remains bounded as $\varepsilon \to 0$ and thus (9) holds.

<u>Step 3</u>. Let $\omega \subset\subset \Omega\backslash\{0\}$. Then $u \in C^{2,1}(\omega \times [0,T))$ with $u(x,0) = 0$ on ω.

<u>Proof</u>. Consider the function $\tilde{u}(x,t)$ defined on $\omega \times (-T,+T)$ by[1]

$$\tilde{u}(x,t) = \begin{cases} u(x,t) & \text{if } 0 < t < T \\ 0 & \text{if } -T < t < 0 \end{cases}$$

so that by Step 2 $\tilde{u} \in L^p_{\ell oc}(\omega \times (-T,+T))$. We claim that

(11) $$\tilde{u}_t - \Delta\tilde{u} + |\tilde{u}|^{p-1}\tilde{u} = 0 \quad \text{in} \quad \mathcal{D}'(\omega \times (-T,+T)) \quad .$$

Indeed let $\phi \in \mathcal{D}(\omega \times (-T,+T))$; we must check that

(12) $$-\iint u\phi_t - \iint u\Delta\phi + \iint |u|^{p-1}u\phi = 0 \quad .$$

Let $\eta(t)$ be any smooth non decreasing function on $\mathbb{R}$ such that

$$\eta(t) = \begin{cases} 1 & \text{for } t \geq 2 \\ 0 & \text{for } t \leq 1 \end{cases}$$

and set $\eta_k(t) = \eta(kt)$.

Since u is a solution of (1) we know that

(13) $$-\iint u(\phi\eta_k)_t - \iint u\Delta(\phi\eta_k) + \iint |u|^{p-1}u\ \phi\eta_k = 0 \quad .$$

In order to deduce (12) it suffices to verify that

(14) $$\iint u\phi(\eta_k)_t \to 0 \quad \text{as} \quad k \to \infty \quad .$$

We have

(15) $$\iint u\phi(\eta_k)_t = \iint u(x,t)[\phi(x,t)-\phi(x,0)](\eta_k)_t + \iint u(x,t)\phi(x,0)(\eta_k)_t \quad .$$

By assumption (5) $\int u(x,t)\phi(x,0)dx \to 0$ as $t \to 0$ and thus

(16) $$\iint u(x,t)\phi(x,0)(\eta_k)_t \to 0 \quad \text{as} \quad k \to \infty \quad .$$

On the other hand, by (8) we see that

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(1) We thank M. S. Baouendi for sugges ing this device which led to a simplification of our oric ·˙ pro .

(17) $\quad |\iint u(x,t)[\phi(x,t) - \phi(x,0)](\eta_k)_t| \le \frac{C}{k} \to 0$ as $k \to \infty$.

Combining (15), (16) and (17) we obtain (14). Therefore (11) is proved. It follows (as in Step 1) that $\tilde{u} \in C^{2,1}(\omega \times (-T,+T))$; in particular $u \in C^{2,1}(\omega \times [0,T))$ and $u(x,0) = 0$ on ω.

Let us summarize; so far, we have shown - without any restriction on p - that any solution of (1) satisfying (5) is smooth on $\Omega \times [0,T)$, except possibly at the point $(x,t) = (0,0)$, and that $u(x,0) = 0$ for $x \neq 0$. It remains to prove that u is smooth near $(0,0)$; the restriction $p > \frac{n+2}{n}$ is now essential.

Step 4. There are constants $C, \rho > 0$ and $0 < T_1 < T$ such that

(18) $\quad |u(x,t)| \le \frac{C}{(|x|^2+t)^{n/2}}$ for $|x| < \rho$ and $0 < t < T_1$.

Proof. Let $\rho > 0$ be such that $B_{2\rho}(0) \subset \Omega$; fix $x^0 \in \mathbb{R}^n$ with $0 < |x^0| < \rho$ and fix $R < |x^0|$. Set

$$G = \{(x,t);\ |x - x^0|^2 < R^2 + t \text{ with } 0 < t < T_1\} .$$

By choosing $T_1 > 0$ small enough we may assume that $G \subset \Omega \times (0,T)$. In the region G we define

$$U(x,t) = \frac{C(R^2+t)^{\theta/2}}{(R^2-r^2+t)^{\theta}}$$

with $\theta = \frac{2}{p-1}$, $r = |x - x^0|$ and C a positive constant. We compute

$$U_t - \Delta U + U^p = \frac{\theta}{2}\,\frac{C(R^2+t)^{\frac{\theta}{2}-1}}{(R^2-r^2+t)^{\theta}} - \frac{4C\theta(\theta+1)r^2(R^2+t)^{\theta/2}}{(R^2-r^2+t)^{\theta+2}}$$

$$- \frac{C(2n+1)\theta(R^2+t)^{\theta/2}}{(R^2-r^2+t)^{\theta+1}} + \frac{C^p(R^2+t)^{\frac{\theta p}{2}}}{(R^2-r^2+t)^{\theta p}} .$$

Note that $\theta p = \theta + 2$ and therefore

(19) $\quad U_t - \Delta U + U^p \ge 0$ holds in G

provided

$$C^{p-1}(R^2+t) \geq 4\theta(\theta+1)r^2 + (2n+1)\theta(R^2-r^2+t) \tag{20}$$

i.e.

$$\begin{cases} C^{p-1} \geq (2n+1)\theta \\ C^{p-1} \geq 4\theta(\theta+1) \end{cases} \tag{21}$$

(it suffices to check (20) at the end points $r = 0$ and $r = \sqrt{R^2+t}$).

We choose C large enough (depending on p and n) so that (21) - and consequently (19) - holds. Clearly

$$u(x,t) \leq U(x,t) \quad \text{if} \quad (x,t) \in \partial G \quad \text{and} \quad 0 \leq t < T_1$$

(recall that $U(x,t) = +\infty$ if $(x,t) \in \partial G$ and $0 < t < T_1$, while $u(x,0) = 0 \leq U(x,0)$). By a standard comparison argument we obtain

$$u \leq U \quad \text{on} \quad G \ .$$

In particular

$$u(x^0,t) \leq U(x^0,t) = \frac{C}{(R^2+t)^{\theta/2}} \ .$$

Since R is any number less than $|x^0|$ we have

$$u(x^0,t) \leq \frac{C}{(|x^0|^2+t)^{\theta/2}} \quad \text{for} \quad |x^0| < \rho \quad \text{and} \quad 0 < t < T_1 \ .$$

Finally since $\theta \leq n$ (i.e. $p \geq \frac{n+2}{n}$) we get

$$u(x^0,t) \leq \frac{C_1}{(|x^0|^2+t)^{n/2}}$$

with $C_1 = C(\rho^2+T_1)^{\frac{n-\theta}{2}}$. We conclude the proof of Step 4 by changing u into $-u$.

Step 5. We have

$$\int_{|x|<\rho} \int_0^{T_1} |u(x,t)|^p dxdt < \infty \ . \tag{22}$$

Proof. An easy computation based on (18) shows that

$$\int_{|x|<\rho} \int_0^{T_1} |u(x,t)|dxdt < \infty \quad . \tag{23}$$

Fix a function $\zeta \in \mathcal{D}(\Omega \times (-T,+T))$ with $0 \leq \zeta \leq 1$, $\zeta = 1$ on $B_\rho(0) \times (0,T_1)$ and set

$$\phi_k(x,t) = \eta_k(|x|^2 + t)\zeta(x,t)$$

(the same function η_k as in Step 3). Since ϕ_k vanishes on a neighborhood of $(0,0)$ we deduce from Steps 1 - 3 that

$$-\iint |u|(\phi_k)_t - \iint |u|\Delta\phi_k + \iint |u|^p\phi_k \leq 0 \tag{24}$$

i.e.

$$\iint |u|^p\phi_k \leq \iint |u|(\phi_k)_t + \iint |u|\Delta\phi_k \quad . \tag{25}$$

Set $D_k = \{(x,t);\ \frac{1}{k} < x^2 + t < \frac{2}{k}\}$. We have

$$(\phi_k)_t = \eta'_k\zeta + \eta_k\,\zeta_t$$

$$\Delta\phi_k = (\Delta\eta_k)\zeta + 2\nabla\eta_k\nabla\zeta + \eta_k\,\Delta\zeta$$

and so

$$|(\phi_k)_t| \leq C \qquad \text{outside} \quad D_k \ , \tag{26}$$

$$|(\phi_k)_t| \leq C(k+1) \qquad \text{on} \quad D_k \ , \tag{27}$$

$$|\Delta\phi_k| \leq C \qquad \text{outside} \quad D_k \ , \tag{28}$$

$$|\Delta\phi_k| \leq C(k+1) \qquad \text{on} \quad D_k \ . \tag{29}$$

Combining (25), (23), (26), (27), (28), (29) we obtain

$$\iint |u|^p\phi_k \leq Ck \iint_{D_k} |u| + C \quad . \tag{30}$$

On the other hand, by Step 4

$$\iint_{D_k} |u| \leq C \iint_{D_k} \frac{dxdt}{(|x|^2+t)^{n/2}} \leq Ck^{n/2} \text{ meas } D_k = \frac{C}{k} \text{ meas } D_1 \quad .$$

Therefore $\iint |u|^p\phi_k$ remains bounded as $k \to \infty$ and (22) follows.

Step 6. u is smooth on $\Omega \times [0,T)$ and $u(x,0) = 0$ on Ω.

Proof. Consider the function $\tilde{u}$ defined on $\Omega \times (-T,+T)$ by

$$\tilde{u}(x,t) = \begin{cases} u(x,t) & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases} .$$

In view of Step 5 we know that $\tilde{u} \in L^p_{\ell oc}(\Omega \times (-T,+T))$. We claim that

(31) $$\tilde{u}_t - \Delta\tilde{u} + |\tilde{u}|^{p-1}\tilde{u} = 0 \quad \text{in } \mathcal{D}'(\omega \times (-T,+T))$$

from which we derive - as in Step 1 - that $\tilde{u} \in C^{2,1}(\Omega \times (-T,+T))$ and so $u \in C^{2,1}(\Omega \times [0,T))$ with $u(x,0) = 0$ on Ω.

Let $\zeta \in \mathcal{D}(\Omega \times (-T,+T))$; we must check that

(32) $$-\iint u\,\zeta_t - \iint u\Delta\zeta + \iint |u|^{p-1}u\,\zeta = 0 \quad .$$

We already know that

(33) $$-\iint u(\phi_k)_t - \iint u\Delta\phi_k + \iint |u|^{p-1}u\,\phi_k = 0$$

where $\phi_k(x,t) = \eta_k(x^2+t)\zeta(x,t)$.

It is therefore sufficient to verify that as $k \to +\infty$

(34) $$\iint u(\eta_k)_t\,\zeta \to 0$$

(35) $$\iint u\,\Delta\eta_k\,\zeta \to 0$$

(36) $$\iint u\,\nabla\eta_k\,\nabla\zeta \to 0 \quad .$$

We have

$$\left|\iint u(\eta_k)_t\,\zeta\right| \le Ck \iint_{D_k}|u|$$

$$\left|\iint u\,\Delta\eta_k\,\zeta\right| \le Ck \iint_{D_k}|u|$$

$$\left|\iint u\,\nabla\eta_k\,\nabla\zeta\right| \le C\sqrt{k} \iint_{D_k}|u| \quad .$$

Finally, by Hölder we get

$$\iint_{D_k}|u| \le \left(\iint_{D_k}|u|^p\right)^{1/p} |\text{meas } D_k|^{\frac{1}{p'}} \quad ,$$

Recall that $|\text{meas } D_k| = \dfrac{C}{k^{\frac{n}{2}+1}}$ and that $\frac{1}{p'}(\frac{n}{2}+1) > 1$ (i.e. $p > \frac{n+2}{n}$);

therefore $k\iint_{D_k}|u| \le C\left(\iint_{D_k}|u|^p\right)^{1/p} \to 0$ (by Step 5).

3. Existence and uniqueness for equations (1) - (3) when $0 < p < \frac{n+2}{n}$.

We assume now for simplicity that $\Omega \subset \mathbb{R}^n$ is a bounded domain with a boundary $\partial\Omega$ of class $C^{2+\alpha}(\alpha > 0)$. Let $0 < p < \frac{n+2}{n}$.

Consider the initial value problem

$$u_t - \Delta u + |u|^{p-1}u = 0 \quad \text{on} \quad \Omega \times (0,\infty) \tag{37}$$

$$u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty) \tag{38}$$

$$u(x,0) = u_0(x) \quad \text{on} \quad \Omega \tag{39}$$

The initial data $u_0(x)$ is a bounded measure on Ω i.e.

$$u_0 \in M(\Omega) = C_0(\bar{\Omega})^* \quad ; \tag{40}$$

where $C_0(\bar{\Omega})$ denotes the space of continuous functions on $\bar{\Omega}$ which vanish on $\partial\Omega$.

Theorem 3. There is a unique function $u \in C^{2,1}(\bar{\Omega} \times (0,+\infty))$ solving (37), (38) and such that

$$\lim_{t\to 0} \int u(x,t)\phi(x)dx = \langle u_0,\phi\rangle \quad \forall\, \phi \in C_0(\bar{\Omega}) \quad . \tag{41}$$

In addition $\int_0^\infty \int_\Omega |u|^p dxdt < \infty$.

Remark 3. The conclusion of Theorem 3 is also valid for some unbounded domains Ω, for example $\Omega = \mathbb{R}^n$.

Remark 4. It is presumably possible to solve (37) - (38) - (39) for some values of $p > \frac{n+2}{n}$ and some measures u_0 less singular than δ (for example a spherical distribution of charges) under some appropriate relation between p and the singular part of u_0.

Let $S(t) = e^{t\Delta}$ denote the contraction semigroup generated in $L^1(\Omega)$ by Δ with zero Dirichlet boundary condition.

Let $0 < T < \infty$ and set $Q = \Omega \times (0,T)$. We shall need the following

Lemma 2. Consider the mapping K defined by

$$\{u_0,f\} \mapsto u = S(t)u_0 + \int_0^t S(t-s)f(s)ds$$

i.e. u is the solution of the linear equation

$$\begin{cases} u_t - \Delta u = f & \text{on } \Omega \times (0,T) \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & . \end{cases}$$

Then K is a compact operator from $L^1(\Omega) \times L^1(Q)$ into $L^q(Q)$ for every $q < \frac{n+2}{n}$.

Proof of Lemma 2. We already know (see Baras [3]) that K is a compact operator from $L^1(\Omega) \times L^1(Q)$ into $L^1(Q)$. Therefore it suffices to check that K is a bounded operator from $L^1(\Omega) \times L^1(Q)$ into $L^q(Q)$ for every $q < \frac{n+2}{n}$.

Recall that for every $1 \leq q \leq \infty$ we have

(42) $$\|S(t)u_0\|_{L^q(\Omega)} \leq \frac{C}{t^{\frac{n}{2}(1-\frac{1}{q})}} \|u_o\|_{L^1(\Omega)} \quad ;$$

inequality (42) follows by Hölder's inequality from the extreme cases $q = 1$, $q = \infty$ (and the case $q = \infty$ is obtained, via the maximum principle from the explicit representation of $e^{t\Delta}$ in $\mathbb{R}^n$).

We deduce from (42) (and Young's inequality) that

$$\|u\|_{L^q(Q)} \leq C(\|u_o\|_{L^1(\Omega)} + \|f\|_{L^1(Q)})$$

provided $q < \frac{n+2}{n}$ (in order for the function $t^{\frac{n}{2}(-1+\frac{1}{q})}$ to lie in $L^q(0,T)$).

Proof of Theorem 3

Existence. Let $u_{0j} \in \mathcal{D}(\Omega)$ be a sequence such that

(43) $$\|u_{0j}\|_{L^1(\Omega)} \leq C$$

(44) $$u_{0j} \to u_o \text{ in the } w^* \text{ topology of } M(\Omega) \quad .$$

Let u_j be the solution of (37) - (38) corresponding to the initial data u_{0j}. One has the following estimates

(45) $$\|u_j\|_{L^\infty(0,T;L^1)} \leq \|u_{0j}\|_{L^1(\Omega)} \leq C$$

(46) $$\int_0^T \int_\Omega |u_j|^p dxdt \leq \|u_{0j}\|_{L^1(\Omega)} \leq C \quad ;$$

indeed, multiply (37) by $\theta_m(u_j)$ where θ_m is a sequence of smooth nondecreasing functions converging to sign. It follows from Lemma 2 that u_j is compact in $L^q(Q)$ for every $q < \frac{n+2}{n}$. We choose a subsequence still denoted by u_j such that $u_j \to u$ in $L^q(Q)$ for every $q < \frac{n+2}{n}$; and thus

(47) $$|u_j|^{p-1}u_j \to |u|^{p-1}u \text{ in } L^1(Q) \quad .$$

On the other hand an easy comparison argument shows that

(48) $$|u_j(\cdot,t)| \leq S(t)\ |u_{0j}| \quad \text{on } Q$$

and therefore

$$\|u_j(\cdot,t)\|_{L^\infty(\Omega)} \leq \frac{C}{t^{n/2}} \|u_{0j}\|_{L^1(\Omega)} \leq \frac{C}{t^{n/2}} \quad .$$

Consequently $u \in L^\infty((\delta,T); L^\infty(\Omega))$ for every $\delta > 0$ and u satisfies

$$u(t) = S(t)u_0 - \int_0^t S(t-s)\ |u(s)|^{p-1}u(s)ds \quad .$$

We conclude - via a standard bootstrap - that $u \in C^{2,1}(\bar\Omega \times (0,T])$ (and in fact u is as smooth as the function $u \mapsto |u|^{p-1}u$ permits). Here $S(t)u_0$ is defined on $M(\Omega)$ as the adjoint of the continuous contraction semigroup $e^{t\Delta}$ on $C_0(\bar\Omega)$; as such $S(t)$ is not a continuous semi-group on $M(\Omega)$ but $S(t)u_0 \to u_0$ in the w^* topology of $M(\Omega)$ as $t \to 0$.

<u>Remark 5</u>. Assume u_0 is an L^1 <u>function</u> instead of a measure. Then, problem (37) - (38) - (39) has a solution for every $0 < p < \infty$. This is a consequence of the Crandall-Liggett Theorem (see [15]) applied in $L^1(\Omega)$ to the m-accretive operator $Au = -\Delta u + |u|^{p-1}u$ (see Brezis-Strauss [11]). The same conclusion can also be obtained directly as follows: let $u_{0j} \in \mathcal{D}(\Omega)$ be

a sequence such that $u_{0j} \to u_0$ strongly in $L^1(\Omega)$. Multiplying (37) by $\theta_m(u_j - u_k)$ we obtain

$$\int |u_j(x,T) - u_k(x,T)|dx + \int_0^T \int_\Omega \left| |u_j|^{p-1}u_j - |u_k|^{p-1}u_k \right| dxdt$$

$$\leq \int |u_{0j}(x) - u_{0k}(x)|dx \to 0 \quad \text{as} \quad j,k \to \infty \quad .$$

Therefore $|u_j|^{p-1}u_j$ is a Cauchy sequence in $L^1(Q)$ and converges strongly in $L^1(Q)$. Thus we have proved (47) without any restriction on p (note that the assumption $p < \frac{n+2}{n}$ enters in the proof of Theorem 3 only in order to obtain (47)).

Uniqueness. Here we need no restriction on p; so let $0 < p < \infty$ be arbitrary. First, observe that if $u \in C^{2,1}(\bar{\Omega} \times (0,T])$ satisfies (37), (38) and (41), then

(49) $$u \in L^1(Q) \quad \text{and} \quad \int_0^T \int_\Omega |u|^p dxdt < \infty$$

and

(50) $$-\int_0^T \int_\Omega u\zeta_t - \int_0^T \int_\Omega u\Delta\zeta + \int_0^T \int_\Omega |u|^{p-1}u\zeta = \langle u_0, \zeta(\cdot,0) \rangle \quad \forall \zeta \in W$$

where

$$W = \{\zeta \in C^{2,1}(\bar{\Omega} \times [0,T]);\ \zeta(x,T) = 0 \ \text{ on } \ \Omega,\ \zeta(x,t) = 0 \ \text{ on } \ \partial\Omega \times [0,T]\} \quad .$$

Indeed from (41) and the uniform boundedness principle we see that $u \in L^\infty(0,T; L^1(\Omega))$. Next, we have for $\varepsilon > 0$

$$\int_\Omega |u(x,T)|dx + \int_\varepsilon^T \int_\Omega |u|^p dxdt \leq \int_\Omega |u(x,\varepsilon)|dx$$

(multiply (37) by $\theta_m(u)$ and integrate over $\Omega \times (\varepsilon,T)$) and thus $\int_0^T \int_\Omega |u|^p dxdt < \infty$.

Finally in order to prove (50) multiply (37) by ζ, integrate on $\Omega \times (\varepsilon,T)$, and pass to the limit as $\varepsilon \to 0$ (notice that $\int u(x,\varepsilon)\zeta(x,\varepsilon)dx \to \langle u_0,\zeta(\cdot,0)\rangle$). We shall now establish uniqueness within the class of function u satisfying (49) - (50). Let u_1, u_2 be two solutions and set $v = u_1 - u_2$. We have

$$- \int_0^T \int_\Omega v(\zeta_t + \Delta\zeta) = \int_0^T \int_\Omega f\zeta \quad \forall\ \zeta \in W$$

where $f = -|u_1|^{p-1}u_1 + |u_2|^{p-1}u_2$. Uniqueness is a direct consequence of the following

Lemma 3. Assume $v \in L^1(Q)$, $f \in L^1(Q)$ satisfy

(51) $$- \int_0^T \int_\Omega v(\zeta_t + \Delta\zeta) = \int_0^T \int_\Omega f\zeta \quad \forall\ \zeta \in W \ .$$

Then

(52) $$\int_0^t \int_\Omega f \text{ sign } v \, dxds \geq \int_\Omega |v(x,t)|dx \quad \text{for all} \quad t \in [0,T] \ .$$

Proof of Lemma 3. Notice that for any given $f \in L^1(Q)$ there is a unique $v \in L^1(Q)$ satisfying (51). Indeed if

$$\int_0^T \int_\Omega v(\zeta_t + \Delta\zeta) = 0 \quad \forall\ \zeta \in W$$

then take ζ such that

$$\zeta_t + \Delta\zeta = h \quad \text{on} \quad \Omega \times (0,T)$$

$$\zeta(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,T)$$

$$\zeta(x,T) = 0 \quad \text{on} \quad \Omega$$

(where $h(x,t)$ is arbitrary and smooth) to deduce that $\int_0^T \int_\Omega vh = 0$.

From the preceding remark on uniqueness it follows that if we solve

(53) $$\begin{cases} \dfrac{\partial v_j}{\partial t} - \Delta v_j = f_i & \text{on} \quad \Omega \times (0,T) \\ v_i(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ v_i(x,0) = 0 & \text{on} \quad \Omega \end{cases}$$

with $f_i \to f$ in $L^1(Q)$, then $v_j \to v$ in $C([0,T]; L^1(\Omega))$. Multiplying (53_1) by $\theta_m(v_j)$ we obtain

$$\int \chi_m(v_j(x,t))dx \leq \int_0^t \int_\Omega f_i \ \theta_m(v_j)dxds$$

where $\chi_m' = \theta_m$. Taking first $j \to \infty$ and then $\theta_m \to$ sign we get (52).

4. The limiting behavior of u_j as $u_{0j} \to \delta$ in case $p > \frac{n+2}{n}$.

We return now to the case $p > \frac{n+2}{n}$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary with $0 \in \Omega$.

Consider a sequence u_j of solutions of (37) - (38) corresponding to a sequence of smooth initial data u_{0j} which converges to δ. Since we know that the limiting initial value problem has no solution (with $u_0 = \delta$), it is interesting to study what happens to the sequence u_j as $j \to \infty$.

Theorem 4. Assume u_{0j} is a sequence in $L^1(\Omega)$ such that

(54) $$\|u_{0j}\|_{L^1(\Omega)} \le C$$

(55) $$u_{0j} \to 0 \quad \text{strongly in } L^1(\Omega \setminus B_r(0)) \text{ for every } r > 0 .$$

Let u_j be the solution of (37) - (38) corresponding to the initial data u_{0j}.

Then $u_j \to 0$ uniformly on $\bar{\Omega} \times [\varepsilon, T]$ for every $\varepsilon > 0$.

Proof. As in the proof of Theorem 3 (existence part) we know that

(56) $$\|u_j\|_{L^\infty(0,T;L^1)} \le C$$

(57) $$\|u_j\|_{L^p(0,T;L^p)} \le C$$

(58) $$\|u_j(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{C}{t^{n/2}} \quad \forall \ t > 0 .$$

From standard linear parabolic estimates we see that

$$\|u_j\|_{C^1(\bar{\Omega}\times[\varepsilon,T])} \le C_\varepsilon \quad \forall \ \varepsilon > 0 .$$

In particular

(59) $$u_j \to u \quad \text{uniformly on } \bar{\Omega} \times [\varepsilon,T] \quad \forall \ \varepsilon > 0$$

with $u \in L^\infty(0,T;L^1) \cap L^p(0,T;L^p)$.

Also we know that $u_j \to u$ in $L^q(Q)$ for every $q < \frac{n+2}{n}$ and in particular

(60) $$u_j \to u \quad \text{in} \quad L^1(Q) \quad .$$

Next we show that

(61) $$|u_j|^{p-1}u_j \to |u|^{p-1}u \quad \text{in} \quad L^1(0,T;L^1(\Omega\setminus B_r(0)) \quad \forall\, r > 0$$

Indeed fix $\zeta \in C^2(\bar{\Omega})$ such that

$$0 < \zeta \leq 1$$

$$\zeta = 1 \quad \text{on} \quad \Omega\setminus B_r(0)$$

$$\zeta = 0 \quad \text{on} \quad B_{r/2}(0) \quad .$$

Multiplying the equation

$$\frac{\partial}{\partial t}(u_j - u_k) - \Delta(u_j - u_k) + |u_j|^{p-1}u_j - |u_k|^{p-1}u_k = 0$$

through by $\zeta\theta(u_j - u_k)$ and letting $\theta \to$ sign we find

$$\int_0^T \int_\Omega \big||u_j|^{p-1}u_j - |u_k|^{p-1}u_k\big|\zeta \leq \int_\Omega |u_{0j} - u_{ok}|\zeta + \int_0^T \int_\Omega |u_j - u_k|\Delta\zeta \quad .$$

Since the right hand side tends to 0 as $j,k \to \infty$ we obtain (61).

As a consequence of (59), (60), (61) we have

(62) $$\int_0^T \int_\Omega u(\zeta_t + \Delta\zeta) + \int_0^T \int_\Omega |u|^{p-1}u\,\zeta = 0$$

for every $\zeta \in W$ such that $\zeta \equiv 0$ near $(0,0)$. Since $u \in L^p(Q)$ and $p > \frac{n+2}{n}$ we deduce as in Step 6 of Section 2 that

(63) $$-\int_0^T \int_\Omega u(\zeta_t + \Delta\zeta) + \int_0^T \int_\Omega |u|^{p-1}u\,\zeta = 0 \quad \forall\, \zeta \in W \quad .$$

We conclude by uniqueness (see the proof of Theorem 3) that $u \equiv 0$.

Remark 6. Assume in addition to (54) - (55) that $u_{0j} \to \delta$ in the w^* topology of $M(\Omega)$. Then we have

(64) $$\int_0^T \int_\Omega |u_j|^{p-1}u_j\,\zeta \to \zeta(0,0) \quad \forall\, \zeta \in C(\bar{Q}) \quad .$$

Indeed let $\zeta \in W$; we have

$$\iint_Q |u_j|^{p-1}u_j\,\zeta = \iint_Q u_j(\zeta_t + \Delta\zeta) + \int_\Omega u_{0j}(x)\zeta(x,0)dx \to \zeta(0,0)$$

since $u_j \to 0$ in $L^1(Q)$ (see (60)). We derive (64) from (59), (61), (57) and a density argument. Notice that (64) is <u>not in contradiction</u> with the fact that $u_j \to 0$ in $L^q(Q)$ for $q < \frac{n+2}{n}$.

<u>Remark 7</u>. The conclusion of Theorem 4 may be viewed as a <u>boundary layer phenomenon</u> at $t = 0$. In the process of passing to the limit, equation (37) has been preserved, as well as the boundary condition (38); however <u>the initial condition has been lost</u>. More generally the argument above shows that if $u_0 \in L^1(\Omega)$ and if u_{0j} is a sequence of initial data such that $\|u_{0j}\|_{L^1(\Omega)} \leq C$ and $u_{0j} \to u_0$ in $L^1(\Omega \setminus B_r(0))$ for every $r > 0$. Then the corresponding solutions u_j converge to u [uniformly on $\bar{\Omega} \times [\varepsilon,T]$, for each $\varepsilon > 0$] where u is the unique solution of (37) - (38) - (39). Again one may lose the "natural" initial condition (for example when $u_{0j} \to u_0 + \delta$ in the w^* topology of $M(\Omega)$ then u takes the initial value u_0).

5. The porous medium equation

Consider the equation

$$u_t - \Delta(|u|^{m-1}u) = 0 \qquad \text{on} \quad \Omega \times (0,T) \tag{65}$$

$$u(x,t) = \qquad \text{on} \quad \partial\Omega \times (0,T) \tag{66}$$

$$u(x,0) = u_0(x) \quad \text{on} \quad \Omega \tag{67}$$

with $0 < m < \infty$.

There is extensive literature dealing with equation (65); see e.g. the expository paper of Peletier [22] and recent contributions by Caffarelli-Friedman [13], [14], Aronson-Benilan [1], Benilan-Crandall [7], Benilan [5], Veron [24], Brezis-Crandall [10], Pierre [23], Crandall-Pierre [16]. The case $m < 1$ corresponds to a "fast diffusion process"; equations of this type appear in plasma problems, see e.g. Berryman-Holland [8].

When $\Omega = \mathbb{R}^n$, $u_0(x) = \delta(x)$ and $m > \frac{n-2}{n}$ (no restriction on m if $n = 1$ or 2) an explicit solution of (65) was found by Barenblatt [4] (see also Pattle [21]), namely

$$u(x,t) = \frac{1}{t^{\ell}}\, G\Big(\frac{|x|}{t^{\ell/n}}\Big)$$

where

$$G(s) = [(\beta^2 - cs^2)^+]^{\frac{1}{m-1}}$$

$c = \frac{\ell(m-1)}{2mn}$, $\ell = \frac{1}{m-1+\frac{2}{n}}$ and β is a positive constant such that

$\int_{\mathbb{R}^n} G(|x|)dx = 1$. A direct calculation shows that $u(x,t) \to \delta(x) \otimes 1(t)$ as $m \downarrow \left(\frac{n-2}{n}\right)$. This suggests that no solution of (65) exists, in the sense of distributions, when $m = \frac{n-2}{n}$ and $u_0 = \delta$ (since one cannot make sense out of δ^m).

We shall now proceed to prove that indeed when $0 < m \le \frac{n-2}{n}$ $(n \ge 3)$ no solution of (65) exists for $u_0 = \delta$. On the other hand when $m > \left(\frac{n-2}{n}\right)$ a solution of (65) exists for any measure u_0.

5.1. Non existence when $0 < m \leq \frac{n-2}{n}$.

Assume $0 < m \leq \frac{n-2}{n}$ $(n \geq 3)$; let $\Omega \subset \mathbb{R}^n$ be any open set with $0 \in \Omega$.

Definition. A strong solution of (65) is a function $u \in L^\infty_{\ell oc}(\Omega)$ such that $u_t \in L^1_{\ell oc}(Q)$ and such that (65) holds in $\mathcal{D}'(Q)$.

Theorem 5. There exists no strong nonnegative solution of (65) such that

$$\text{(68)} \qquad \operatorname*{ess\,lim}_{t\to 0} \int u(x,t)\phi(x)dx = \phi(0) \quad \forall \phi \in C_c(\Omega) .$$

Remark 8. It is reasonable to believe that there is no weak solution of (65) (i.e. a function $u \in L^1_{\ell oc}(Q)$ such that (65) holds in $\mathcal{D}'(Q)$) satisfying (68).

Theorem 5 is a direct consequence of

Theorem 6. Let u be a strong solution of (65) such that

$$\text{(69)} \qquad \operatorname*{ess\,lim}_{t\to 0} \|u(\cdot,t)\|_{L^1(\omega)} = 0 \quad \forall \omega \subset\subset \Omega\setminus\{0\}.$$

Then

$$\text{(70)} \qquad \operatorname*{ess\,lim}_{t\to 0} \|u(\cdot,t)\|_{L^1(B_r(0))} = 0 \quad \text{for some } r > 0 .$$

Proof of Theorem 6.

Let $0 < \rho < 1$ be such that $B_{2\rho}(0) \subset \Omega$. Let $x^0 \in \mathbb{R}^n$ with $0 < |x^0| < \rho$. Let $0 < R < |x^0|$ and set

$$V(x) = \frac{C\, R^{n-2}}{(R^2 - |x-x^0|^2)^{n-2}} \quad \text{for } x \in B_R(x^0) .$$

V is a positive smooth function in $B_R(x^0)$ and $V = \infty$ on $\partial B_R(x^0)$. The same computation as in Brezis-Veron [12] shows that for some appropriate positive constant C (depending only on n) one has

$$\text{(71)} \qquad -\Delta V + V^p \geq 0 \quad \text{on } B_R(x^0),\ \forall p \geq \frac{n}{n-2} .$$

Set $p = \frac{1}{m}$, $\lambda = \frac{1}{1-m}$ and

$$\text{(72)} \qquad U(x,t) = t^\lambda V^p(x) \quad \text{on } B_R(x^0) \times (0,\infty) .$$

It follows from (71) that

$$(73) \qquad U_t - \Delta U^m \geq 0 \quad \text{on} \quad B_R(x^0) \times (0,\infty) \quad .$$

Also

$$(74) \qquad U(x,t) = \infty \quad \text{on} \quad \partial B_R(x^0) \times (0,\infty)$$

$$(75) \qquad U(x,0) = 0 \quad \text{on} \quad B_R(x^0) \quad .$$

By comparison of (65) and (73) we shall deduce that

$$(76) \qquad u \leq U \quad \text{on} \quad B_R(x^0) \times (0,T) \quad .$$

Indeed, Kato's inequality - which is valid since u and U are strong solutions - asserts that

$$\Delta(|u|^{m-1}u - |U|^{m-1}U)^+ \geq [\Delta(|u|^{m-1}u - |U|^{m-1}U)]\operatorname{sign}^+(|u|^{m-1}u - |U|^{m-1}U)$$

and

$$\frac{\partial}{\partial t}(u - U)^+ = \frac{\partial}{\partial t}(u - U)\operatorname{sign}^+(u - U) \quad .$$

Since $\operatorname{sign}^+(|u|^{m-1}u - |U|^{m-1}U) = \operatorname{sign}^+(u - U)$ we conclude that

$$(77) \quad \frac{\partial}{\partial t}(u - U)^+ - \Delta(|u|^{m-1}u - |U|^{m-1}U)^+ \leq 0 \quad \text{in} \quad \mathcal{D}'(B_R(x^0) \times (0,T)) \quad .$$

On the other hand $(|u|^{m-1}u - |U|^{m-1}U)^+ \equiv 0$ in a neighborhood of $\partial B_R(x^0) \times (\varepsilon, T-\varepsilon)$.

Thus by integrating (77) we find, for $\varepsilon < t < T-\varepsilon$,

$$(78) \qquad \int_{B_R(x^0)} (u(x,t) - U(x,t))^+ dx \leq \int_{B_R(x^0)} (u(x,\varepsilon) - U(x,\varepsilon))^+ dx \quad .$$

As $\varepsilon \to 0$, the right hand side in (78) tends to 0 (by assumption (69)) and (76) is proved. Similarly we obtain $|u| \leq U$ on $B_R(x^0) \times (0,T)$ and in particular $|u(x^0,t)| \leq U(x^0,t) = \frac{Ct^\lambda}{R^{(n-2)p}}$. Since $R < |x^0|$ is arbitrary we have

$$|u(x^0,t)| \leq \frac{Ct^\lambda}{|x^0|^{(n-2)p}} \quad \text{on} \quad B_\rho(0) \times (0,T)$$

and therefore

$$(79) \qquad |u(x,t)|^m \leq C\,\frac{t^{m\lambda}}{|x^0|^{n-2}} \quad \text{on} \quad B_\rho(0) \times (0,T) \quad .$$

Finally we claim that

$$(80) \qquad \int_{B_{\rho/2}} |u(x,t)|dx \leq C\,t^\lambda$$

which proves (70).

Indeed, by Kato's inequality we have

$$\frac{\partial}{\partial t}|u| - \Delta|u|^m \le 0 \quad \text{in} \quad \mathcal{D}'(\Omega) \quad . \tag{81}$$

Fix a smooth function $\phi(x)$, $0 \le \phi \le 1$ with support in $B_\rho(0)$ such that $\phi = 1$ on $B_{\rho/2}(0)$.

Let η_k be a sequence of functions as in Step 3 of Section 2. Multiplying (81) by $\phi(x)\eta_k(|x|)$ we find

$$\int_\Omega |u(x,t)|\phi(x)\eta_k(|x|)dx \le \int_0^t \int_\Omega |u|^m \Delta(\phi\eta_k)dxds =$$

$$= \int_0^t \int_\Omega |u|^m(\eta_k\Delta\phi + 2\nabla\eta_k\nabla\phi + \Delta\eta_k\phi)dxdx$$

$$\le C\int_0^t \int_{B_\rho(0)} |u|^m dxds + C(k+k^2)\int_0^t \int_{\frac{1}{k}<|x|<\frac{2}{k}} |u|^m dxds \quad .$$

Using (79) we find that

$$\int_\Omega |u(x,t)|\phi(x)\eta_k(|x|)dx \le Ct^\lambda \quad .$$

We obtain (80) by letting $k \to \infty$.

5.2. <u>Existence when</u> $m > \frac{n-2}{n}$.

Assume (for simplicity) that $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary. Let $m > \frac{n-2}{n}$ (any $m > 0$ if $n = 1$ or 2).

<u>Theorem 7</u>. For every $u_0 \in M(\Omega)$ there exists a function $u(x,t)$ satisfying

$$u \in C((0,T]; L^1) \cap L^\infty(0,T; L^1) \cap L^\infty(\Omega \times (\varepsilon,T)) \;\forall \varepsilon > 0 \quad , \tag{82}$$

$$|u|^m \in L^1(Q) \quad , \tag{83}$$

$$-\iint u\zeta_t - \iint |u|^{m-1}u\Delta\zeta = \langle u_0, \zeta(\cdot,0)\rangle \;\forall \zeta \in W^{(1)} \quad . \tag{84}$$

---

(1) Recall that
$W = \{\zeta \in C^{2,1}(\bar\Omega \times [0,T];\ \zeta(x,T) = 0 \text{ on } \Omega,\ \zeta(x,t) = 0 \text{ on } \partial\Omega \times [0,T]\}$

In particular we have

(85) $$\lim_{t\to 0} \int_{\Omega} u(x,t)\phi(x)dx = \langle u_0,\phi\rangle \quad \forall\, \phi \in C_0(\bar{\Omega}) \quad .$$

Remark 9. When $\Omega = \mathbb{R}^n$, $m > 1$ and $u_0 \geq 0$ an existence and uniqueness result has been obtained by Pierre [23] for the equation (65) - (66) - (67). We suspect that under the assumptions of Theorem 7 the solution is also unique.

Remark 10. It is presumably possible to solve problem (65) - (66) - (67) for some values of $0 < m \leq \frac{n-2}{n}$ and some measures u_0 less singular than δ (for example a spherical distribution of changes) under some appropriate relation between m and the singular part of u_0.

Proof of Theorem 7.

We denote by $S(t)$ the L^1 contraction semigroup generated by $\Delta(|u|^{m-1}u)$ via the Crandall-Liggett Theorem. We recall some properties of $S(t)$:

i) $S(t)$ is smoothing from L^1 into L^∞. More precisely we have

(86) $$\|S(t)u_0\|_{L^\infty(\Omega)} \leq [\frac{C}{t} \|u_0\|_{L^1(\Omega)}^{\frac{2}{n}}]^k, \;\forall\, t > 0, \text{ with } k = (m-1 + \frac{2}{n})^{-1} \;;$$

see Benilan [5] (and also Veron [24]).

ii) $S(t)$ is compact in L^1; that is, for each fixed $t > 0$, $S(t)$ maps L^1-bounded sets into L^1-compact sets, see Baras [3].

iii) The mapping $u_0 \mapsto \{S(t)u_0\}_{0<t<T}$ maps L^1 bounded sets into compact subsets of $L^1(Q)$, see Baras [3].

Given $u_0 \in M(\Omega)$ we consider a sequence u_{0j} of smooth functions such that $\|u_{0j}\|_{L^1} \leq C$ and $u_{0j} \to u_0$ in the w^* topology of $M(\Omega)$. Set $u_j = S(t)u_{0j}$ so that

(87) $$\|u_j(\cdot,t)\|_{L^1(\Omega)} \leq C$$

$$\|u_j(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{C}{t^k} \quad \forall\, t > 0 \tag{88}$$

$$u_j \to u \quad \text{in} \quad C((0,T];\, L^1) \tag{89}$$

$$u_j \to u \quad \text{in} \quad L^1(Q) \tag{90}$$

with u satisfying (82).

Next, we deduce from Hölder's inequality, (87) and (88) that

$$\|u_j(\cdot,t)\|_{L^q(\Omega)} \le \frac{C}{t^{k(1-\frac{1}{q})}} \quad \forall \quad 1 \le q \le \infty \tag{91}$$

and therefore

$$\|u_j\|_{L^q(Q)} \le C \quad \text{provided} \quad q < m + \frac{2}{n} \ . \tag{92}$$

In particular we derive from (90) and (92) that

$$u_j \to u \quad \text{in} \quad L^q(Q) \quad \text{for every} \quad q < m + \frac{2}{n} \ ; \tag{93}$$

thus

$$|u_j|^{m-1}u_j \to |u|^{m-1}u \quad \text{in} \quad L^1(Q) \ . \tag{94}$$

Using (90) and (94) we obtain (84).

Finally we show that (84) implies (85). Indeed in (84) choose $\zeta(x,t) = \phi(x)\eta(t)$ with $\phi \in C^2(\bar{\Omega})$, $\phi = 0$ on $\partial\Omega$ and $\eta \in C^1([0,T])$ with $\eta(T) = 0$.

Setting $g(t) = \int_\Omega u(x,t)\phi(x)dx$ and $h(t) = \int_\Omega |u|^{m-1}u\Delta\phi dx$ we have

$$g \in L^\infty(0,T) \cap C((0,T]), \quad h \in L^1(0,T)$$

and by (84),

$$-\int_0^T g(t)\eta'(t)dt - \int_0^T h(t)\eta(t)dt = \langle u_0,\phi\rangle\, \eta(0) \quad \forall\, \eta \in C^1([0,T]) \ .$$

Consequently $\lim_{t\to 0} g(t) = \langle u_0,\phi\rangle$, that is

$$\lim_{t\to 0} \int u(x,t)\phi(x)dx = \langle u_0,\phi\rangle \quad \forall\, \phi \in C^2(\bar{\Omega}) \cap C_0(\bar{\Omega}) \ .$$

We derive (85) using a density argument and the fact that $u \in L^\infty(0,T,L^1)$.

5.3. <u>The limiting behavior of</u> u_j <u>as</u> $u_{0j} \to \delta$ <u>in case</u> $m < \frac{n-2}{n}$.

We return now to the case $0 < m < \frac{n-2}{n}$ $(n \geq 3)$.

Let $\Omega \subset \mathbf{R}^n$ be <u>either</u> a bounded domain with smooth boundary <u>or</u> $\Omega = \mathbf{R}^n$.

<u>Theorem 8</u>. Assume u_{0j} is a sequence in $L^1(\Omega)$ such that $u_{0j} \to \delta$ in the w^* topology of $M(\Omega)$ and that $\operatorname{Supp} u_{0j} \subset B_{1/j}(0)$.

Let u_j be the (semi-group) solution of (65) - (66) corresponding to the initial data u_{0j}.

Then $u_j(x,t) \to \delta(x) \otimes 1(t)$ in the w^* topology of $M(Q)$.

<u>Proof</u>

<u>Step 1</u>. Assume $\Omega = \mathbf{R}^n$, $u_{0j} \geq 0$, $\|u_{0j}\|_{L^1} \leq C$ and $\operatorname{Supp} u_{0j} \subset B_{1/j}(0)$. Then

(95) $$u_j(x,t) \to 0 \quad \text{a.e. on } \mathbf{R}^n \times (0,T) \quad .$$

Indeed, by the techniques of Section 5.1 we obtain

(96) $$|u_j(x,t)| \leq \frac{Ct^\lambda}{|x|^{(n-2)p}} \quad \text{for} \quad |x| > \frac{2}{j},\ t > 0$$

(notice that in the present context comparison is not a difficulty since u_j is the semi group solution; therefore u_j is obtained by some limiting procedure and the comparison can be made at each step of the approximation).

Thus

(97) $$|u_j(x,t)|^m \leq \frac{Ct^{\lambda m}}{|x|^{n-2}} \quad \text{for} \quad |x| > \frac{2}{j},\ t > 0 \quad .$$

Next we claim that

(98) $$\int_{\frac{4}{j}<|x|<4j} |u_j(x,t)|dx \leq Ct^\lambda \quad \text{for} \quad t > 0 \quad .$$

Indeed we have for every $\phi \in \mathcal{D}(\mathbf{R}^n)$

(99) $$\int_{\mathbf{R}^n} u_j(x,t)\phi(x)dx = \int_{\mathbf{R}^n} u_j(x,0)\phi(x)dx + \int_0^t \int_{\mathbf{R}^n} u_j^m(x,s)\Delta\phi(x)dxds \quad .$$

We choose ϕ in such a way that

$$\begin{cases} \phi(x) = 0 \quad \text{for} \quad |x| < \frac{2}{j} \quad \text{and for} \quad |x| > 8j \\ \phi(x) = 1 \quad \text{for} \quad \frac{4}{j} < |x| < 4j \\ |\Delta\phi| \leq Cj^2 \quad \text{for} \quad \frac{2}{j} < |x| < \frac{4}{j} \\ |\Delta\phi| \leq \frac{C}{j^2} \quad \text{for} \quad 4j < |x| < 8j \quad . \end{cases}$$

Then, we derive (98) from (97) and (99). Next, we extract a subsequence - still denoted by u_j such that $u_j(x,t)$ <u>converges to some limit</u> $u(x,t)$ <u>a.e. on</u> Q.

This is justified as follows. Let $\phi \in \mathcal{D}_+(\mathbb{R}^n\setminus\{0\})$. Multiplying (formally - but this can be justified) (65) by $u_j^{2-m}\phi$ we obtain

$$\frac{1}{3-m}\int u_j^{3-m}(x,t)\phi(x)dx + (2-m)m\int_0^t\int |\nabla u_j|^2\phi dxdx$$

$$= \frac{1}{3-m}\int u_j^{3-m}(x,0)\phi(x)dx + \frac{m}{2}\int_0^t\int u_j^2\Delta\phi \quad .$$

If j is large enough - so that $\operatorname{Supp}\phi \cap B_{2/j}(0) = \emptyset$ - we see, using (96), that $\int_0^t\int |\nabla u_j|^2\phi dxds \leq C$. Therefore (u_j) is compact in $L^2(\omega \times (0,T))$ for $\omega \subset\subset \mathbb{R}^n\setminus\{0\}$ (by Aubin's compactness Lemma, see e.g. J. L. Lions [20]). The limit u satisfies

$$u(x,t) \leq \frac{Ct^\lambda}{|x|^{(n-2)p}} \quad \text{a.e. on} \quad \mathbb{R}^n \times (0,T) \tag{100}$$

$$\int u(x,t)dx \leq Ct^\lambda \quad \text{for a.e.} \quad t \quad . \tag{101}$$

Since $u_j \to u$ in $L^1(\omega \times (0,T))$ for $\omega \subset\subset \mathbb{R}^n\setminus\{0\}$, the function u also verifies

$$\frac{\partial u}{\partial t} - \Delta u^m = 0 \quad \text{in} \quad \mathcal{D}'((\mathbb{R}^n\setminus\{0\}) \times (0,T)) \quad . \tag{102}$$

The same argument as in Section 5.1 leads from (102) to

$$\frac{\partial u}{\partial t} - \Delta u^m = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n \times (0,T)) \quad . \tag{103}$$

[Use the sequence $\eta_k(|x|)$ and notice that by Hölder,

$$k^2 \int_0^t \int_{\frac{1}{k}<|x|<\frac{2}{k}} u^m \le k^2\left(\int_0^t \int_{\frac{1}{k}<|x|<\frac{2}{k}} u\right)^m (k^{-n}t)^{1-m} \to 0 \quad \text{as} \quad k \to \infty] \quad .$$

Therefore

$$(104) \qquad \frac{\partial}{\partial t}(E*u) + u^m = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n \times (0,T))$$

where $E*u = (-\Delta)^{-1}u = \dfrac{C_n}{|x|^{n-2}} * u$.

We conclude from (101) and (104) that $\frac{\partial}{\partial t}(E*u) \le 0$ and consequently $E*u \equiv 0$; thus $u \equiv 0$.

<u>Step 2</u>. Proof of Theorem 8 concluded in the general case.

From Step 1 we deduce that $u_j(x,t) \to 0$ a.e.

Indeed, by comparison we have

$$|u_j| \le S(t)\,|u_{0j}|$$

where $S(t)$ denotes the semi group generated in <u>$L^1(\mathbb{R}^n)$</u> by $\Delta|u|^{m-1}u$; by Step 1 we know that $S(t)|u_{0j}| \to 0$ a.e. on $\mathbb{R}^n \times (0,T)$.

We have for every $\zeta \in \mathcal{D}(\Omega \times [0,T])$

$$-\iint u_j \frac{\partial\zeta}{\partial t} - \iint |u_j|^{m-1}u_j\Delta\zeta = \langle u_{0j}, \zeta(\ ,0)\rangle \quad .$$

Since $|u_j|^{m-1}u_j \to 0$ in $L^1(Q)$ we obtain at the limit

$$(104) \qquad -\iint u_j \frac{\partial\zeta}{\partial t} \to \zeta(0,0) \quad \forall\, \zeta \in \mathcal{D}(\Omega \times [0,T)) \quad .$$

Given $\theta \in \mathcal{D}(\Omega \times (0,T))$ we set

$$\zeta(x,t) = \int_t^T \theta(x,s)ds$$

and we find

$$\iint u_j\theta \to \int_0^T \theta(0,s)ds = \langle\delta(x) \otimes 1(t), \theta\rangle \quad \forall\, \theta \in \mathcal{D}(\Omega \times (0,T)) \quad .$$

Since u_j is bounded in $L^1(Q)$ we conclude by density that $u_j(x,t) \to \delta(x) \otimes 1(t)$ in the w^* topology of $M(Q)$.

<u>Remark 11</u>. The two essential ingredients in the proof of existence (Theorem 7), namely the $L^1 \to L^\infty$ smoothing and the L^1 compactness of $S(t)$ <u>fail</u> when $0 < m \le \frac{n-2}{n}$. This is a clear consequence of Theorem 8. Another view

point is the following. Consider in a bounded domain Ω the L^1 m-accretive operator $Au = -\Delta(|u|^{m-1}u)$ with zero Dirichlet boundary condition. Its resolvent $J_\lambda = (I + \lambda A)^{-1} (\lambda > 0)$ is <u>not compact</u> in $L^1(\Omega)$; this follows from the fact that the equation $-\Delta u + |u|^{p-1}u = \delta$ has no solution when $p > \frac{n}{n-2}$, see Brezis-Veron [12]. On the other hand it is easy to show that J_λ maps <u>bounded</u> sets from any $L^q(\Omega)$, $q > 1$ into <u>compact</u> sets of $L^1(\Omega)$.

We deduce that:

i) $S(t)$ is <u>not compact</u> in $L^1(\Omega)$; indeed when a semi-group $S(t)$ is compact, then the resolvent J_λ is also compact, see Brezis [9].

ii) $S(t)$ is <u>not smoothing</u> from $L^1(\Omega)$ into any $L^q(\Omega)$, $q > 1$. Suppose, by contradiction, that there is a $q > 1$ such that

(105) $\|S(t)u_0\|_{L^q(\Omega)} \leq C(t) \quad \forall\, t \in (0,T),\ \forall\, u_0 \in L^1$ with $\|u_0\|_{L^1} \leq M$.

From the regularizing effect of Benilan-Crandall [7] we know that

$$\|J_\lambda\, S(t)u_0 - S(t)u_0\|_{L^1} \leq \frac{C\lambda}{t} \quad \text{where } C = \frac{2\|u_0\|_{L^1}}{|m-1|} .$$

It follows that $S(t)$ is compact in $L^1(\Omega)$. Indeed <u>fix</u> $0 < t < T$ and <u>fix</u> $\varepsilon > 0$; set $\lambda = \frac{t\varepsilon}{2C}$. By assumption (105) the set $C = \{S(t)u_0;\ \|u_0\|_{L^1} \leq M\}$ is bounded in $L^q(\Omega)$ and so the set $D = \{J_\lambda\, S(t)u_0;\ \|u_0\|_{L^1} \leq M\}$ is compact in L^1. Therefore the set D (resp. C) may be covered by a finite collection of balls of radius $\frac{\varepsilon}{2}$ (resp. ε) in $L^1(\Omega)$.

The preceding argument shows nevertheless that $S(t)$ enjoys two compactness properties:

a) $S(t)$ maps bounded sets from any $L^q(\Omega)$, $q > 1$, into compact sets of $L^1(\Omega)$.

b) $S(t)$ maps bounded sets from $L^1(\Omega)$ into compact sets of $L^q(\Omega)$ for any $0 < q < 1$.

[The lack of regularizing effect of $S(t)$ from L^1 into L^q for any $q > 1$ when $m < \frac{n-2}{n}$ had been obtained earlier by Benilan and Crandall in $\Omega = \mathbb{R}^n$ using a simple homogeneity argument.]

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We first consider the Cauchy problem for

(1) $$u_t - \Delta u + |u|^{p-1} u = 0 \quad \text{on} \quad \Omega \times (0,T)$$

with a boundary condition and the initial condition

(2) $$u(x,0) = \delta(x) \quad \text{on} \quad \Omega$$

where $\Omega \subset \mathbb{R}^n$ is domain containing 0, $0 < p < \infty$, $0 < T < \infty$ and $\delta(x)$ is the Dirac mass at 0. We prove that a solution of (1) - (2) exists <u>if and only if</u>

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ABSTRACT (continued)

$0 < p < \frac{n+2}{n}$. When $0 < p < \frac{n+2}{n}$ we actually prove a more general existence and uniqueness result in which (2) is replaced by

(3) $$u(x,0) = u_0(x) \quad \text{on} \quad \Omega$$

where u_0 is a <u>measure</u>.

Next, we discuss the Cauchy problem for

(4) $$u_t - \Delta(|u|^{m-1}u) = 0 \quad \text{on} \quad \Omega \times (0,T)$$

where $0 < m < \infty$, with a boundary condition and the initial condition (3). We prove that a solution of (4) - (2) exists if and only if $m > \frac{n-2}{n}$. When $m > \frac{n-2}{n}$ we actually prove existence for the problem (4) - (3).

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